Algebra Preliminary Examination

June, 2015

Do all of the following questions.

Question 1. Prove Schur's Lemma. Let G be a finite group with V and W irreducible complex G-representations.

1. If V and W are not isomorphic, then $\mathsf{Hom}_G(V, W) = 0$.
2. If V and W are isomorphic, then $\mathsf{Hom}_G(V, W) \cong \mathbb{C}$.

Question 2. Consider the alternating group A_4 of even symmetries of the set $\{a, b, c, d\}$. The group A_4 has 4 conjugacy classes. These are represented by the unit 1, an order 2 element $(ab)(cd)$, the order 3 element (abc), and the order 3 element (acb). Calculate the character table of A_4. Construct the irreducible representations of A_4. Prove your answers are correct.

Question 3. Consider $\mathsf{GL}_n(\mathbb{F}_q)$ the general linear group of $n \times n$ invertible matrices over the field with $q = p^r$ elements, and let $B_n \subset \mathsf{GL}_n(\mathbb{F}_q)$ be the subgroup of upper triangular matrices which have 1's on the diagonal.

1. Derive a formula for the number of elements of $\mathsf{GL}_n(\mathbb{F}_q)$.
2. Prove that $B_n \subset \mathsf{GL}_n(\mathbb{F}_q)$ is a Sylow p-subgroup.

Question 4. Consider the polynomial

$$f(x) = x^5 - 8x + 2 \ .$$

Prove that $f(x)$ is irreducible over $\mathbb{Q}$. Prove that the Galois group of $f(x)$ is Σ_5.

Question 5. Let A be a commutative ring with

$$M \xrightarrow{\phi} M$$

an A-linear endomorphism of finitely generated A-module M. Prove that if ϕ is surjective, then it is also injective. (Hint: recall Nakayama's Lemma, and note that ϕ gives M the structure of an module over the polynomial ring $A[x]$.)

Question 6. Consider the commutative ring

$$A := \mathbb{C}[s, t]/(t^3 - s^2) \ .$$

1. Prove that A is an integral domain.
2. Calculate the integral closure of A, and prove that your answer is correct.
3. Classify the maximal ideals of A according to the ranks of their cotangent spaces.